INVARIANT BILEAR FORMS AND CASIMIR OPERATOR FOR F.D. SIMPLE $\mathfrak{g}$.

KAC - MOODY CASE.

A A SYMMETRIC GENERALIZED CARTAN (SYMMETRIZABLE SIMILAR.)

$\mathfrak{g} = \mathfrak{g}(A) = \mathfrak{h} \oplus \mathfrak{n}_- \oplus \mathfrak{n}_+$

$\mathfrak{n}_+ = \langle e_i, i = 1, \dots, r \rangle$

$\mathfrak{n}_- = \langle f_i, i = 1, \dots, r \rangle$

$\mathfrak{h}$ HAS ELEMENTS $\alpha_i^\vee$ $\qquad \mathfrak{g}^* \quad \alpha_i$

LINEARLY INDEPENDENT

$$\mathfrak{h} = \bigoplus \mathbb{C}\alpha_i^\vee \oplus \mathfrak{h}''$$

THEOREM: $\mathfrak{g}$ HAS AN INVARIANT SYMMETRIC BILINEAR FORM SUCH THAT:

$$(\alpha_i^\vee | \alpha_j^\vee) = a_{ij}$$

$\nu : \mathfrak{h} \to \mathfrak{h}^* \qquad (H | H') = \langle H, \nu(H') \rangle$

THIS IS AN ISOMORPHISM. TRANSFER THE SCALAR PRODUCT TO $\mathfrak{h}^*$: $(\lambda | \mu) = (\nu^{-1}(\lambda) | \nu^{-1}(\mu))$

START BY DEFINING $(\ |\ )$ ON $\mathfrak{g}$.

$\mathfrak{h} = \bigoplus \mathbb{C}\alpha_i^\vee \oplus \mathfrak{h}''$ ⟵ CHOICE TO BE MADE.

DETERMINED ON THIS PIECE BY

$(\alpha_i^\vee | \alpha_j^\vee) = a_{ij}$.

$$(\alpha_i^\vee | H) = \langle \underset{\mathfrak{h}}{H}, \underset{\mathfrak{h}^*}{\gamma(\alpha_i^\vee)} \rangle$$

CONSISTENT SINCE $A = (a_{ij})$ IS SYMMETRIC.

DEFINES $(x|y)$ IF EITHER x OR y IS IN $\sum \mathbb{C}\alpha_i^\vee$. EXTEND IT TO ALL $\mathfrak{h}$ BY $(\mathfrak{h}''|\mathfrak{h}'') = 0$.

AD INVARIANCE REQUIRES;

$$([x, y] | z) = (x | [y, z]) \checkmark$$

OR $([x, y] | z) = -(y | [x, z])$. ✓

PROVED TUESDAY:

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$$

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid HX = \alpha(H) X \text{ FOR } H \in \mathfrak{h} \}$$

$$\underset{\mathfrak{h}}{\langle X}, \underset{\mathfrak{h}^*}{\alpha \rangle} = \alpha(X)$$

IF $\alpha = 0$, $\mathfrak{g}_\alpha = \mathfrak{h}$.

IF $\alpha \neq 0$, α IS A ROOT $\mathfrak{g}_\alpha = \mathfrak{X}_\alpha$.

PROPOSITION: IF $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$ AND $(,)$ IS AN AD-INVARIANT BILINEAR FORM x, y ARE ORTHOGONAL UNLESS $\alpha = -\beta$.

LET US DEFINE A GRADING ON $\mathfrak{g}$.

$$\mathfrak{g}_0 = \mathfrak{h}$$

$\mathfrak{g}_1$ = SPAN OF e_i, $\mathfrak{g}_{-1}$ = SPAN OF f_i

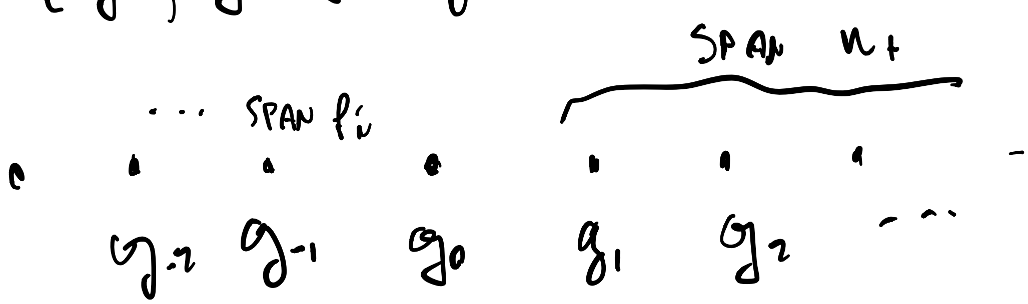
$\mathfrak{g}_n$ = SPAN OF $[e_{i_1}, [e_{i_2} \cdots [e_{i_{n-1}}, e_{i_n}]]$

$i_1 \leq \cdots \leq i_n$.

$$[g_i, g_j] \subseteq \bigoplus_{n \leq i+j} g_n.$$

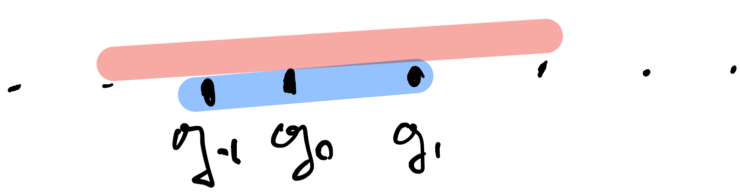
$$g^{(N)} = \bigoplus_{n \leq N} g_n \quad \text{FILTRATION}.$$

$$[g_i, g^{(N)}] \subset g^{(N+|i|)}.$$



$i, j > 0 \quad [g_i, g_j] \subset \bigoplus_{n = i+j} g_n$

IDEA: FIRST DEFINE $(\, , \,)$ ON $g^{(1)}$



THEN SHOW IF DEFINED ON $g^{(N)}$ EXTEND TO $g^{(N+1)}$.

CLAIM: THERE IS A WEAKLY AD INVARIANT PAIRING ON $\mathfrak{g}^{(1)} = \mathfrak{g}_{(-1)} \oplus \mathfrak{h}_0 \oplus \mathfrak{g}_1$ SUCH THAT IF $x, y, z, [x,y], [y,z] \in \mathfrak{g}^{(1)}$ THEN

$$([x,y] \mid z) = (x \mid [x,z]).$$

CLAIM: IF SUCH A PAIRING EXISTS ON $\mathfrak{g}^{(n)}$ IT CAN BE EXTENDED TO $\mathfrak{g}^{(n+1)}$.

FOR $(\mid)$ ON $\mathfrak{g}^{(1)} = \mathfrak{h} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ WE REQUIRE $(x \mid y) = 0$ IF $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, $\alpha \neq -\beta$. WE REQUIRE $(,)$ BE THE ONE ON $\mathfrak{h}$ DEFINED ABOVE BY THE CARTAN MATRIX. THIS LEAVES US TO DEFINE $(x \mid y)$ IF $x \in \mathfrak{X}_\alpha$, $y \in \mathfrak{X}_{-\alpha}$. ($\alpha \in \Delta^+$.)

(AT THIS STAGE α WOULD BE A SIMPLE ROOT α_i SO WLOG $x = e_i$, $y = f_i$.)

WE ACTUALLY WANT $[x, y] = (x \mid y) \nu^{-1}(\alpha)$.

CHOOSE $(\cdot|\cdot)$ SO $(e_i|f_i)=1$.

$[e_i, f_i] = \alpha_i^\vee$.

THIS IS ONE OF THE DEFINING RELATION OF $\mathfrak{g}(A)$. NEED TO SHOW IF

IN $\mathfrak{g}(A)$ $\left\{ \begin{array}{l} x, y, z \text{ IN SOME } \mathfrak{g}_\alpha \text{ WHERE} \\ \alpha \text{ EITHER } 0 \text{ OR A SIMPLE ROOT OR} \\ \text{ITS NEGATIVE}. \end{array} \right.$

THEN I WANT TO KNOW

$$([x, y] | z) = (x | [y, z]).$$

IN EVERY CASE BOTH SIDES ARE ZERO EXCEPT:

$$([e_i, f_i] | H) = (e_i | [f_i, H]) \qquad h \in \mathfrak{h}$$

BOTH SIDES EQUAL $\alpha_i(H)$

LHS $= (\alpha_i^\vee | H) = \alpha_i(H)$

$[\mathfrak{h}, \mathfrak{g}_i]$.

RHS $= (e_i \mid \alpha_i(H) f_i) = \alpha_i(H)(e_i \mid f_i)$

$$([e_i, H] \mid f_i) = (e_i \mid [H, f_i])$$

BOTH SIDES $= \alpha_i(H)(e_i \mid f_i)$

$$[e_i, H] = -[H, e_i] = \alpha_i(H) e_i .$$

$$([f_i, h] \mid e_i) = (f_i \mid [h, e_i])$$

BOTH SIDES $= \alpha_i(H)$.

CLAIM: IF SUCH A PAIRING EXISTS ON $\mathfrak{g}^{(n)}$ IT CAN BE EXTENDED TO $\mathfrak{g}^{(n+1)}$.

ASSUME IF $x, y, z, [x, y], [y, z] \in \mathfrak{g}^{(n)}$

$$([x, y] \mid z) = (x \mid [y, z])$$

TO EXTEND TO $\mathfrak{g}^{(n+1)}$ WRITE $y \in \mathfrak{g}^{(n+1)}$

$$y = \sum [u_i, v_i] .$$

DEFINE $(x \mid y) := \sum_i ([x, u_i] \mid v_i)$

WHY IS THIS WELL-DEFINED ?

CLEARLY WELL DEFINED AS A FN OF x.

NOT CLEAR A FN OF y.

WRITE $x = \sum_j [s_j, t_j]$ $\qquad s_j, t_j \in \mathfrak{g}^{(n)}$.

I WILL SHOW

$$\sum_i ([x, y_i] \mid v_i) = \sum_j (s_j \mid [t_j, y])$$

SECOND EXPRESSION SHOWS y INVARIANCE

$$([[s_j, t_j], u_i \mid v_i) =$$
$$(s_j \mid [t_j [u_i\, v_i]])\,.$$

$$(x \mid y) = \sum_i ([x, u_i] / v_i)$$
$$= \sum_{i,j} ([[s_j, t_j], u_i] \mid v_i)\,.$$

USING JACOBI IDENTITY

$$([[s_j\, t_j]\, u_i \mid v_i) =$$

$$([[s_j\, u_i]\, t_j] \mid v_i) + ([s_j\, [t_j\, u_i]] \mid v_i)$$

$$= \underbrace{([s_j\, u_i] \mid [t_j\, v_i])}_{(s_j \mid [u_i\, [t_j\, v_i]])} + (s_j \mid [[t_j\, u_i]\, v_i])$$

TAKES PLACE IN $\mathfrak{g}^{(n)}$

$$= (s_j \mid [t_j\, [u_i\, v_i]])$$

ANOTHER APPLICATION OF JACOBI.

IT REMAINS TO CHECK INVARIANCE BUT THIS BUILT INTO THE CONSTRUCTION.

WE PROVED:

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THIS IS AN ISOMORPHISM. TRANSFER THE SCALAR PRODUCT TO $\mathfrak{h}^*$: $(\lambda | \mu) = (\nu^{-1}(\lambda) | \nu^{-1}(\mu))$

IT IS ALSO TRUE IF $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ $\alpha \in \Delta^+$ THEN

$$(x | y) = [x, y] \nu^{-1}(\alpha).$$

$\nu : \mathfrak{h} \to \mathfrak{h}^*$ ISO.

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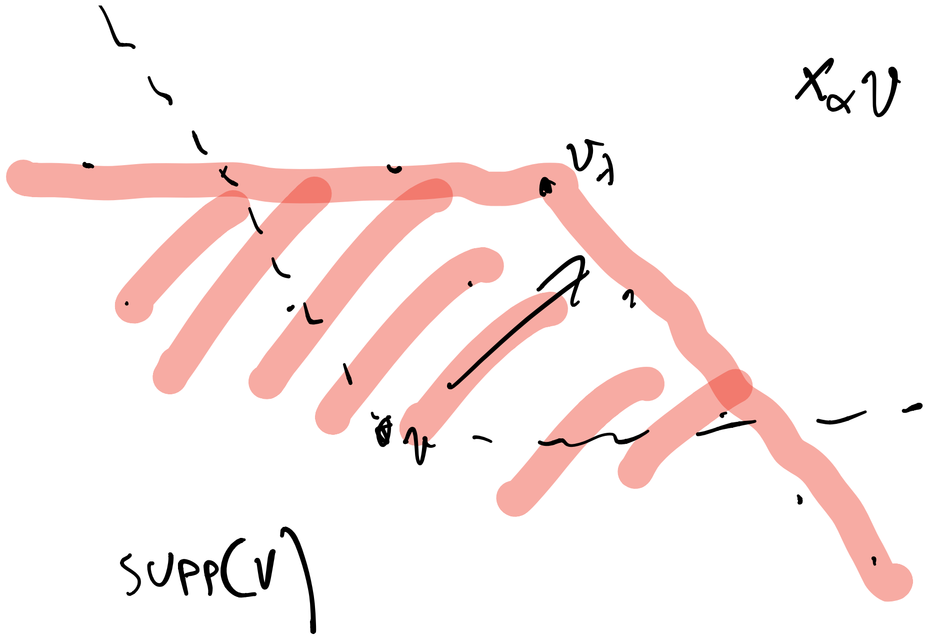

$x_\alpha v$
v_λ
λ
v
supp(v)